

## A generalization of d'Alembert formula

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**Abstract.** In this paper we find a closed form of the solution for the factored inhomogeneous linear equation

$$\prod_{j=1}^n \left( \frac{d}{dt} - A_j \right) u(t) = f(t).$$

Under the hypothesis  $A_1, A_2, \dots, A_n$  are infinitesimal generators of mutually commuting strongly continuous semigroups of bounded linear operators on a Banach space  $X$ . Here we do not assume that  $A_j$ s are distinct and we offer the computational method to get explicit solutions of certain partial differential equations.

**Keywords.** d'Alembert formula;  $C_0$ -semigroup.

### 1. Introduction

Many homogeneous differential equations can be transformed to factored equations of the form:

$$\begin{cases} \prod_{j=1}^n \left( \frac{d}{dt} - A_j \right) u(t) = 0, \\ u^{(k)}(0) = x_k \in X, k = 0, 1, 2, \dots, n-1, \end{cases} \quad (1.1)$$

where  $A_1, A_2, \dots, A_n$  are infinitesimal generators of mutually commuting strongly continuous semigroups of bounded linear operators on a Banach space  $X$  and  $u^{(k)}(0)$  is the  $k$ th derivative of  $u(t)$  at  $t = 0$ . Under the hypotheses,

- (A0)  $A_1, A_2, \dots, A_n$  generate  $C_0$ -semigroups on a Banach space  $X$  which are mutually commuting. That is  $e^{tA_j}e^{sA_k} = e^{sA_k}e^{tA_j}$  for all  $t, s \geq 0$  and for all  $j, k \in \{1, 2, \dots, n\}$ .
- (A1)  $A_j - A_k$  is injective if  $j \neq k$ ;
- (A2) range  $(A_j - A_k)$  is large enough for  $j \neq k$ .

Goldstein *et al* [2] found the solution of (1.1) by the d'Alembert formula, which has the form

$$u(t) = \sum_{j=1}^n e^{tA_j} x_j. \quad (1.2)$$

One may say  $u(t)$  is either a strong or mild solution of (1.1) and we ignore this issue for the moment. Recently, the abstract d'Alembert formula has been broadly applied to equipartition of energy and scattering theory (see for e.g. [3,4,5]); it also has been extended to semigroups that are not strongly continuous (see [6]). However in these papers, the authors always assume that all  $A_j$ s are different. They directly showed that the function  $u(t)$  given by (1.2) is a solution of (1.1) by putting  $u(t)$  into the differential equation in their papers. This motivated us to consider the case that some of the  $A_j$ s in the abstract factored linear equation (1.1) are equal. We get a unique solution of (1.1) by a constructive way. The most interesting point is that one may easily follow this process to get the explicit form of certain differential equation (see §4).

Throughout this paper we always assume that hypothesis (A0) holds. Under this assumption  $A_1, A_2, \dots, A_n$  are mutually commuting, therefore permuting the orders of the operators in (1.1) will not influence the solution. Thus, one may permute operators in (1.1) such that the same operators put together as

$$(P1) \quad A_1 = A_2 = \dots = A_{S_1} = B_1, A_{S_1+1} = A_{S_1+2} = \dots = A_{S_1+S_2} = B_2, \dots$$

$$A_{(\sum_{j=1}^{i-1} S_j)+1} = \dots = A_{\sum_{j=1}^i S_j} = B_i \quad \text{and} \quad \sum_{j=1}^i S_j = n.$$

With these notations we assume that

(A1)'  $B_j - B_k$  is injective if  $j \neq k$ ;

(A2)' range  $(B_j - B_k)$  is large enough for  $j \neq k$ .

Furthermore, to the inhomogeneous initial value problem

$$\prod_{j=1}^n \left( \frac{d}{dt} - A_j \right) u(t) = f(t), \quad f(0) \neq 0, \\ u^{(k)}(0) = x_k, k = 0, 1, 2, \dots, n-1, \quad (1.3)$$

we also assume that

(H1)  $f \in C^1([0, T]; X) \cap (\cup_{i=1}^n C[0, T]; [D(A_i)])$ , where  $[D(A_i)]$  is the Banach space  $D(A_i)$  equipped with the graph norm.

Under these assumptions we have the following results.

**Theorem 1.** Suppose the assumptions (A0), (A1)' and (A2)' are all fulfilled, then there exists an unique solution of the homogeneous initial value problem (1.1) which can be expressed as

$$u(t) = \sum_{k_1=0}^{S_1-1} \frac{t^{k_1}}{k_1!} T_{B_1}(t) y(n, k_1) + \sum_{k_2=0}^{S_2-1} \frac{t^{k_2}}{k_2!} T_{B_2}(t) y(n, S_1 + k_2) \\ + \dots + \sum_{k_i=0}^{S_i-1} \frac{t^{k_i}}{k_i!} T_{B_i}(t) y \left( n, \left( \sum_{j=1}^{i-1} S_j \right) + k_i \right), \quad (1.4)$$

where  $S_j$  is the multiplicity of  $B_j$ ,  $\sum_{j=1}^i S_j = n$ ,  $\{T_{B_j}(t)\}_{t \geq 0}$  is the  $C_0$ -semigroup generated by  $B_j$  and the relation between the coefficient vector

$$\vec{y} = (y(n, 0), y(n, 1), \dots, y(n, n-1))^T,$$

and the initial data vector

$$\vec{x} = (u(0), u'(0), \dots, u^{(n-1)}(0))^T$$

can be represented as  $M_n^{-1}\vec{x} = \vec{y}$ . Here the matrix  $M_n$  is composed by the sub-matrices  $(B_j)_{S_j}$ ,  $j = 0, 1, 2, \dots, i$ , that is

$$M_n = [(B_1)_{S_1} \ (B_2)_{S_2} \ \dots \ (B_i)_{S_i}] \quad (1.5)$$

and the sub-matrix  $(B_j)_{S_j}$  is a  $n \times S_j$  matrix which is formed by the first  $S_j$  columns of the matrix

$$(B_j)_n = \begin{bmatrix} I & 0 & & & \\ B_j & I & 0 & & \\ B_j^2 & 2B_j & I & & \\ \vdots & \vdots & \vdots & \ddots & 0 \\ B_j^{n-1} & C_{n-2}^{n-1} B_j^{n-2} & C_{n-3}^{n-1} B_j^{n-3} & \dots & I \end{bmatrix}. \quad (1.6)$$

**Theorem 2.** Suppose the assumptions  $(A1)'$ ,  $(A2)'$  and  $(H1)$  are fulfilled,  $u^{(k)}(0) = 0$  ( $k = 0, 1, 2, \dots, n-1$ ) and  $f(0) \neq 0$ , the nontrivial solution of the inhomogeneous initial value problem (1.4) can be expressed as

$$\begin{aligned} u(t) = & \sum_{k_1=0}^{S_1-1} \int_0^t \frac{1}{k_1!} (t-s)^{k_1} T_{B_1}(t-s) Z(n, k_1) f(s) \, ds \\ & + \sum_{k_2=0}^{S_2-1} \int_0^t \frac{1}{k_2!} (t-s)^{k_2} T_{B_2}(t-s) Z(n, S_1 + k_2) f(s) \, ds + \dots \\ & + \sum_{k_i=0}^{S_i-1} \int_0^t \frac{1}{k_i!} (t-s)^{k_i} T_{B_i}(t-s) Z\left(n, \sum_{j=1}^{i-1} S_j + k_i\right) f(s) \, ds, \end{aligned} \quad (1.7)$$

where  $S_i$  is the multiplicity of  $B_i$  and  $\sum S_i = n$ ,  $\{T_{B_i}(t)\}_{t \geq 0}$  is the  $C_0$ -semigroup generated by  $B_i$ . Furthermore, if  $\vec{h} = (0, 0, \dots, I)$ ,  $I$  is the identity operator on  $C^1([0, T]; X) \cap (\cup_{i=1}^n C[0, T]; [D(A_i)])$ ,  $\vec{z} = (Z(n, 0), Z(n, 1), \dots, Z(n, n-1))^T$  is a vector and  $M_n = [(B_1)_{S_1} (B_2)_{S_2} \dots (B_i)_{S_i}]$  is the matrix defined as in Theorem 1 and the following relation holds:

$$M_n^{-1} \vec{h} = \vec{z}. \quad (1.8)$$

Furthermore, if any one of the initial data of (1.3) does not equal to zero, then the nontrivial solution of the inhomogeneous initial value problem (1.3) can be obtained by combining the results of Theorems 1 and 2 (see Corollary 3).

## 2. Homogeneous equation

For proving Theorem 1, we will use the following three lemmas as preliminaries. The proof of them are either straightforward or can be found in ref. [8], which we omit here.

*Lemma 1.* Let  $u_{j+1}(t) = \prod_{k=1}^j \left(\frac{d}{dt} - A_k\right)u_1(t)$ ,  $j = 1, 2, \dots, n-1$  for all  $t \geq 0$  and assume that  $u_j(t) \in D(A_j)$  for all  $t \geq 0$ . Then (1.1) is equivalent to the vector-valued initial value problem

$$\begin{cases} \frac{d\vec{u}(t)}{dt} = \begin{bmatrix} A_1 & 1 & & & \\ & A_2 & 1 & & \\ & & \ddots & & \\ & 0 & & A_{n-1} & 1 \\ & & & & A_n \end{bmatrix} \vec{u}(t), \\ \vec{u}(0) = (u_1(0), u_2(0), \dots, u_n(0)). \end{cases} \quad (2.1)$$

where  $\vec{u}(t) \equiv (u_1(t), u_2(t), \dots, u_n(t))$ . The initial data component element in (2.1) is

$$u_m(0) = u_m^* = x_{m-1} + \sum_{k=0}^{m-2} (-1)^k \sum (A_{i_{j_1(k)}} \cdots A_{i_{j_k(k)}} x_{m-k}), \quad m = 1, 2, \dots, n \quad (2.2)$$

with  $i_{j_s(k)} < i_{j_t(k)}$  if  $s < t$  for all  $i_{j_s(k)} \in \{1, 2, \dots, n\}$  and  $x_m$  is the initial data in (1.1).

*Lemma 2.* As long as  $x \in D(A_i) \cap D(A_j)$ ,

$$\int_0^t T_i(t-s)T_j(s)x ds = (A_j - A_i)^{-1}(T_i(t) - T_j(t))x ds$$

for all  $1 \leq i, j \leq n$  and for all  $0 < s \leq t$ . Furthermore, for any integer  $k \geq 2$ ,

$$\begin{aligned} \int_0^t T_i(t-s) \frac{s^k}{k!} T_j(s)x ds &= \frac{t^k}{k!} (A_j - A_i)^{-1} T_j(t)x \\ &\quad - \frac{1}{(k-1)!} (A_j - A_i)^{-1} \int_0^t T_i(t-s) s^{k-1} T_j(s)x ds \end{aligned}$$

for all  $1 \leq i, j \leq n$  and for all  $0 < s \leq t$ .

*Lemma 3.* Let  $X$  be a reflexive Banach space and let  $A$  be the infinitesimal generator of a  $C_0$ -semigroup  $\{T(t)\}_{t \geq 0}$  on  $X$ . If  $f$  is a Lipschitz continuous function on  $[0, T]$ , then for every  $x \in D(A)$  the initial value problem

$$\begin{cases} \frac{d}{dt}u(t) = Au(t) + f(t) \\ u(0) = x \end{cases} \quad (2.3)$$

has a unique solution on  $[0, T]$  given by

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds.$$

(see p. 109 of [8]).

*Proof of Theorem 1.* To show that this theorem is true, we prove two special cases at first.

*Case 1.* Suppose

$$\prod_{j=1}^n \left( \frac{d}{dt} - A_j \right) u(t) = \left( \frac{d}{dt} - A \right)^n u(t) = 0 \quad (2.4)$$

(that is  $A_j = A = B_1$  for all  $j = 1, 2, \dots, n$ ). By Lemma 1, solving (1.1) is equivalent to solving (2.1). One may find the solution of (2.1) by successively solving  $u_k(t)$ ,  $k = n, n-1, \dots, 1$ . In fact  $u_n$  is the solution of the initial value problem

$$\begin{cases} \left( \frac{d}{dt} - A \right) u_n(t) = 0 \\ u_n(0) = u_n^* \end{cases},$$

where  $u_n^*$  is defined by (2.2). Then  $u_n(t) = T(t)u_n^*$  where  $\{T(t)\}_{t \geq 0}$  is the  $C_0$ -semigroup generated by  $A$ . One may get  $u_{n-1}(t)$  from  $u_n(t)$  by solving the follow equation:

$$\begin{cases} \frac{d}{dt} u_{n-1}(t) = A u_{n-1}(t) + u_n(t) \\ u_{n-1}(0) = u_{n-1}^* \end{cases},$$

where  $u_{n-1}^*$  is defined by (2.2).

In fact,

$$\begin{aligned} u_{n-1}(t) &= T(t)u_{n-1}^* + \int_0^t T(t-s)u_n(s) \, ds \\ &= T(t)u_{n-1}^* + \int_0^t T(t-s)T(s)u_n^* \, ds \\ &= T(t)u_{n-1}^* + tT(t)u_n^* = T(t)y(2,0) + tT(t)y(2,1), \end{aligned}$$

where  $y(2,0) = u_{n-1}^*$  and  $y(2,1) = u_n^*$ . In general, if we get

$$u_{n-k}(t) = \sum_{j=0}^k \frac{t^j}{j!} T(t)y(k+1, j) \quad \text{for } k = 0, 1, 2, \dots, n-1,$$

then one may get  $u_{n-(k+1)}(t)$  by solving the initial value problem

$$\begin{cases} \left( \frac{d}{dt} - A \right) u_{n-(k+1)}(t) = u_{n-k}(t) = \sum_{j=0}^k \frac{t^j}{j!} T(t)y(k+1, j); \\ u_{n-(k+1)}(0) = u_{n-(k+1)}^*. \end{cases}$$

The solution of this initial value problem is

$$\begin{aligned} u_{n-(k+1)}(t) &= T(t) u_{n-(k+1)}(0) + \int_0^t T(t-s) \sum_{j=0}^k \frac{t^j}{j!} T(s)y(k+1, j) \, ds \\ &= T(t) u_{n-(k+1)}(0) + \sum_{j=0}^k \frac{t^{j+1}}{(j+1)!} T(t)y(k+1, j). \end{aligned}$$

For simplifying the notation, we denote

$$u_{n-(k+1)}(t) = \sum_{j=0}^{k+1} \frac{t^j}{j!} T(t) y(k+2, j), \quad (2.5)$$

where  $y(k+2, 0) = u_{n-(k+1)}(0)$  and  $y(k+2, j)$  in (2.5) which is equal to  $y(k+1, j-1)$  in (2.4) for all  $j = 1, 2, \dots, k$ . Although, the expression of vector  $y(n, k)$  is not very clear till now, we will find the expression  $y(n, k)$  in terms of initial values  $u^{(k)}(0)$ s at the end of this proof.

Case 2. Suppose eq. (1.1) is expressed with some suitable initial data as

$$\left( \frac{d}{dt} - B \right) \left( \frac{d}{dt} - A \right)^{n-1} u(t) = 0. \quad (2.6)$$

As in Case 1, one can get the solution of the equation

$$\left( \frac{d}{dt} - A \right)^{n-1} u(t) = 0. \quad (2.7)$$

Denote the solution of (2.7) by  $u_2(t) = \sum_{k=0}^{n-2} \frac{t^k}{k!} T_A(t) y(n-1, k)$ , where  $\{T_A(t)\}_{t \geq 0}$  is the  $C_0$ -semigroup generated by  $A$ .

Then solving the initial value problem (2.6) is equivalent to solving the following initial value problem

$$\begin{cases} \left( \frac{d}{dt} - B \right) u_1(t) &= u_2(t); \\ u_1(0) &= u_1^*. \end{cases} \quad (2.8)$$

By Lemma 2,

$$\begin{aligned} u_1(t) &= T_B(t) u_1^* + \int_0^t T_B(t-s) u_2(s) \, ds \\ &= T_B(t) u_1^* + \int_0^t T_B(t-s) \sum_{k=0}^{n-2} \frac{s^k}{k!} T_A(s) y(n-1, k) \, ds \\ &= T_B(t) u_1^* + \int_0^t T_B(t-s) T_A(s) y(n-1, 0) \, ds + \cdots \\ &\quad + \int_0^t T_B(t-s) \frac{s^k}{k!} T_A(s) y(n-1, k) + \cdots \\ &\quad + \int_0^t T_B(t-s) \frac{s^{(n-2)}}{(n-2)!} T_A(s) y(n-1, n-2) \, ds \\ &\quad \vdots \\ &= T_B(t) u_1^* + \sum_{k=1}^{n-2} \frac{t^k}{k!} (A-B)^{-1} T_A y(n-1, k) \\ &\quad + \sum_{k=2}^{n-2} (-1)^1 \frac{t^{k-1}}{(k-1)!} (A-B)^{-2} T_A y(n-1, k) + \cdots \end{aligned}$$

$$\begin{aligned}
& + (-1)^{n-3} t (A - B)^{-(n-2)} T_A y(n-1, n-2) \\
& + \sum_{k=0}^{n-1} (-1)^k (A - B)^{-(k+1)} (T_A - T_B) y(n-1, k),
\end{aligned}$$

where  $\{T_B(t)\}_{t \geq 0}$  is  $C_0$ -semigroup generated by  $B$ .

Rewrite  $u_1$  in terms of increasing degree of  $t$ , one may get

$$\begin{aligned}
u_1(t) = & T_B(t) \left[ u_1(0) + \sum_{k=0}^{n-2} (-1)^{k+1} (A - B)^{-(k+1)} y(n-1, k) \right] \\
& + T_A(t) \left[ \sum_{k=0}^{n-2} (-1)^k (A - B)^{-(k+1)} y(n-1, k) \right] \\
& + t T_A(t) \left[ \sum_{k=1}^{n-2} (-1)^{k-1} (A - B)^{-k} y(n-1, k) \right] + \cdots \\
& + \frac{t^j}{j!} T_A(t) \left[ \sum_{k=j}^{n-2} (-1)^{k-j} (A - B)^{-(k+1-j)} y(n-1, k) \right] + \cdots \\
& + \frac{t^{n-1}}{(n-1)!} T_A(t) [(-1)^0 (A - B)^{-1} y(n-1, n-2)].
\end{aligned} \tag{2.9}$$

For simplifying the notation, we denote

$$\begin{aligned}
u_1(t) = & T_B(t) [y(n, 0)] + T_A(t) [y(n, 1)] + t T_A(t) [y(n, 2)] + \cdots \\
& + \frac{t^k}{k!} T_A(t) [y(n, k+1)] + \cdots + \frac{t^{n-1}}{(n-1)!} T_A(t) [y(n, n-1)],
\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}
[y(n, 0)] &= \left[ u_1^* + \sum_{k=0}^{n-2} (-1)^{k+1} (A - B)^{-(k+1)} y(n-1, k) \right], \\
[y(n, 1)] &= \left[ \sum_{k=0}^{n-2} (-1)^k (A - B)^{-(k+1)} y(n-1, k) \right], \\
[y(n, j)] &= \left[ \sum_{k=j}^{n-2} (-1)^{k-j} (A - B)^{-(k+1-j)} y(n-1, k) \right] \\
& \quad (2 \leq j \leq n-2) \quad \text{and}
\end{aligned}$$

$$[y(n, n-1)] = [(-1)^{2(n-1)} (A - B)^{-1} y(n-1, n-2)].$$

For the general case, one may first permute operators in (1.1) such that the same operators are put together and then apply the results in Cases 1 and 2 alternately. Finally, one can reach the conclusion that the solution of (1.1) can be expressed as in (1.4). This theorem will be proved as long as one finds the relation between initial data vector

$\vec{x} = (u(0), u'(0), \dots, u^{n-1}(0))^T$  and the vector  $\vec{y} = (y(n, 0), y(n, 1), \dots, y(n, n-1))^T$ . Since the solution  $u(t)$  can be expressed as the combination of the terms

$$u(t; k, l) = \frac{t^k}{k!} e^{tA_l} x, \quad x \in D(A)$$

we consider the derivatives of  $u(t; k, l)$ . From the fact  $C_m^n = C_{m-1}^{n-1} + C_m^{n-1}$ , one may get the  $i$ -th derivative of  $u(t; k, l)$  as

$$\begin{aligned} u(t; k, l)^{(i)} &= \frac{1}{(k-i)!} t^{(k-i)} e^{tA_l} x + \frac{1}{(k-(i-1))!} C_1^i t^{(k-(i-1))} A_l e^{tA_l} x + \dots \\ &\quad + \frac{1}{(k-(i-j))!} C_j^i t^{(k-(i-j))} A_l^j e^{tA_l} x + \dots \\ &\quad + \frac{1}{k!} C_i^i t^k A_l^i e^{tA_l} x \quad \text{for } i < k \end{aligned} \quad (2.11)$$

and

$$\begin{aligned} u(t; k, l)^{(k+j)} &= C_j^{k+j} A_l^j e^{tA_l} + \frac{1}{1!} C_{j+1}^{k+j} A_l^{j+1} t e^{tA_l} + \dots + \frac{1}{h!} C_{j+h}^{k+j} A_l^{j+h} t^h e^{tA_l} + \dots \\ &\quad + \frac{1}{k!} C_{k+j}^{k+j} A_l^{k+j} t^k e^{tA_l} \quad \text{for } j = i - k \geq 0. \end{aligned} \quad (2.12)$$

Since we want to find the relation between the coefficient vector  $\vec{y}$  and initial data vector  $\vec{x}$ , we need only to consider the special situation  $t = 0$ . According to (2.11) and (2.12), if one just considers a single operator  $A_i$ , one can get

$$(A_i)_n = \begin{bmatrix} I & 0 & & & \\ A_i & I & 0 & & \\ A_i^2 & 2A_i & I & & \\ \vdots & \vdots & \vdots & \ddots & 0 \\ A_i^{n-1} & C_{n-2}^{n-1} A_i^{n-2} & C_{n-3}^{n-1} A_i^{n-3} & \dots & I \end{bmatrix}. \quad (2.13)$$

In general, after we permute the operators in (1.1) such that the same operators are put together as (P1), the equation can be represented as

$$\prod_{i=1}^{S_1} \left( \frac{d}{dt} - B_1 \right) \prod_{i=1}^{S_2} \left( \frac{d}{dt} - B_2 \right) \dots \prod_{i=1}^{S_i} \left( \frac{d}{dt} - B_i \right) u = 0$$

and one can find the sub-matrices  $M_n^j$  corresponding to  $\prod_{i=1}^{S_j} \left( \frac{d}{dt} - B_j \right)$  as (1.6). Combining these  $i$  sub-matrices together, one may get  $M_n$  as (1.5). The uniqueness of the solution follows from Lemmas 1 and 3 immediately.



*Remark.* We use the following example to demonstrate how to get the matrix  $M_n$  in Theorem 1. Consider the initial value problem

$$\begin{cases} \left(\frac{d}{dt} - A\right) \left(\frac{d}{dt} - A\right) \left(\frac{d}{dt} - B\right) \left(\frac{d}{dt} - B\right) \left(\frac{d}{dt} - C\right) u(t) = 0; \\ u^{(k)}(0) = x_k, \quad (k = 0, 1, 2, 3, 4). \end{cases} \quad (2.14)$$

By comparing with the proof of theorem 1, one may rewrite the differential equation and initial data in (2.14) as

$$u_5(t) = \left(\frac{d}{dt} - A\right) \left(\frac{d}{dt} - B\right) \left(\frac{d}{dt} - B\right) \left(\frac{d}{dt} - C\right) u(t)$$

and

$$\begin{aligned} u_5^* = u_5(0) = & x_4 - (A + 2B + C)x_3 + (B^2 + 2AB + 2BC + AC)x_2 \\ & - (AB^2 + CB^2 + 2ABC)x_1 + AB^2Cx_0. \end{aligned}$$

At first, solve the initial value problem

$$\begin{cases} \left(\frac{d}{dt} - A\right) u_5(t) = 0 \\ u_5(0) = u_5^*. \end{cases}$$

One may have

$$u_5(t) = T_A(t)u_5^* = T_A(t)y(1, 0),$$

where  $\{T_A(t)\}_{t \geq 0}$  is  $C_0$ -semigroup generated by  $A$ . If we rewrite eq. (2.14) as Lemma 1 and find the solution by successively solving  $u_k(t)$  for  $k = 4, 3, 2$ , then we get

$$u_2(t) = T_By(4, 0) + tT_By(4, 1) + T_Ay(4, 2) + tT_Ay(4, 3).$$

The coefficient vector of  $u_2(t)$  is  $\vec{y}_4 = (y(4, 0), y(4, 1), \dots, y(4, 3))^T$ . By a similar method, we get the coefficient vector of  $u(t) = u_1(t)$  as  $\vec{y}_5 = (y(5, 0), y(5, 1), \dots, y(5, 4))^T$  since  $\vec{y}_4$  is just a 4-dimensional vector. For vector of  $\vec{y}_5$ , we need to extend  $\vec{y}_4$  as a 5-dimensional vector. We add  $u(0)$  into  $\vec{y}_4$  as the first component of  $\vec{y}_4^*$ , that is  $\vec{y}_4^* = (u(0), y(4, 0), y(4, 1), \dots, y(4, 3))^T$ .

Rewrite (2.10) in the matrix form

$$\begin{bmatrix} y(5, 0) \\ y(5, 1) \\ y(5, 2) \\ y(5, 3) \\ y(5, 4) \end{bmatrix} = \begin{bmatrix} I & -(B-C)^{-1} & (B-C)^{-2} & -(A-C)^{-1} & (A-C)^{-2} \\ 0 & (B-C)^{-1} & -(B-C)^{-2} & 0 & 0 \\ 0 & 0 & (B-C)^{-1} & 0 & 0 \\ 0 & 0 & 0 & (A-C)^{-1} & -(A-C)^{-2} \\ 0 & 0 & 0 & 0 & (A-C)^{-1} \end{bmatrix} \begin{bmatrix} u(0) \\ y(4, 0) \\ y(4, 1) \\ y(4, 2) \\ y(4, 3) \end{bmatrix}. \quad (2.15)$$

From Theorem 1, the relation between  $\bar{y}_4^* = (u(0), y(4, 0), y(4, 1), \dots, y(4, 3))^T$  and  $\bar{u}' = (u(0), u_2(0), u_2'(0), u_2''(0), u_2^{(3)}(0))^T$  can be represented as

$$\begin{bmatrix} u(0) \\ y(4, 0) \\ y(4, 1) \\ y(4, 2) \\ y(4, 3) \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & I & 0 \\ 0 & B & I & A & I \\ 0 & B^2 & 2B & A^2 & 2A \\ 0 & B^3 & 3B^2 & A^3 & 3A^2 \end{bmatrix}^{-1} \begin{bmatrix} u(0) \\ u_2(0) \\ u_2'(0) \\ u_2''(0) \\ u_2^{(3)}(0) \end{bmatrix}. \quad (2.16)$$

Followed from Lemma 1, the relation between the initial data of  $u(t)$  and  $u_2(t)$  can be expressed as

$$\begin{bmatrix} u(0) \\ u'(0) \\ u''(0) \\ u^{(3)}(0) \\ u^{(4)}(0) \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ -C & I & 0 & 0 & 0 \\ 0 & -C & I & 0 & 0 \\ 0 & 0 & -C & I & 0 \\ 0 & 0 & 0 & -C & I \end{bmatrix}^{-1} \begin{bmatrix} u(0) \\ u_2(0) \\ u_2'(0) \\ u_2''(0) \\ u_2^{(3)}(0) \end{bmatrix}. \quad (2.17)$$

According to (2.15), (2.16) and (2.17), we get

$$M_5 = \begin{bmatrix} I & I & 0 & I & 0 \\ C & B & I & A & I \\ C^2 & B^2 & 2B & A^2 & 2A \\ C^3 & B^3 & 3B^2 & A^3 & 3A^2 \\ C^4 & B^4 & 4B^3 & A^4 & 4A^3 \end{bmatrix}.$$

### 3. Inhomogeneous equation

In this section, we consider the inhomogeneous initial value problem

$$\begin{cases} \prod_{j=1}^n \left( \frac{d}{dt} - A_j \right) u(t) = f(t), & f(0) \neq 0, \\ u^{(k)}(0) = x_k, k = 0, 1, 2, \dots, n-1. \end{cases} \quad (3.1)$$

According to superposition principle, one may obtain the solution of (3.1) by combining the solution of homogeneous initial value problem (1.1) with nonzero initial data and the solution of inhomogeneous case with zero initial data.

*Proof of Theorem 2.* As in the proof of Theorem 1, we consider  $A_j = A$  for all  $j = 1, 2, \dots, n$  at first. In this case, one may follow the process shown in Theorem 1 to get

$$v_{n-k}(t) = \int_0^t \frac{1}{k!} (t-s)^k T_A(t-s) Z(n, k) f(s) ds, \quad (k = 0, 1, 2, \dots, n-1), \quad (3.2)$$

where  $v_k(t)$  ( $k = 0, 1, 2, \dots, n-1$ ) is the solution of the initial value problem

$$\begin{cases} \left( \frac{d}{dt} - A \right) v_{n-k}(t) = v_{n-(k-1)}(t) \\ v_{n-k}(0) = 0 \end{cases}$$

with  $v_n(t) = f(t)$ . Finally, one may obtain  $v_1(t)$  to be the solution of (3.1) with zero initial data. However, for finding the operators  $Z(n, k)$ s in the representation of (1.7), one may rewrite the initial value problem (3.1) (with zero initial data) in the equivalent system form

$$\begin{cases} \frac{d\vec{u}(t)}{dt} = \begin{bmatrix} A_1 & 1 & & & \\ & A_2 & 1 & & \\ & & \ddots & 0 & \\ & 0 & & A_{n-1} & 1 \\ & & & & A_n \end{bmatrix} \vec{u}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ f \end{bmatrix}; \\ \vec{u}(0) = \vec{u}_0(0) = (u_1(0), u_2(0), \dots, u_n(0)) = (0, 0, \dots, 0), \end{cases} \quad (3.3)$$

where  $u_1(t)$  is the solution of the initial value problem (3.1) with zero initial data and

$$u_{j+1}(t) = \prod_{k=1}^j \left( \frac{d}{dt} - A_k \right) u_1(t) \in D(A_{j+1}) \text{ for all } t \geq 0, \quad 0 \leq j < n.$$

We will show that relation (1.8) holds at the end of this theorem.

Further, we consider  $A_1 = B, A_2 = A_3 = \dots = A_n = A$  (i.e. there are only two distinct operators in eq. (3.1)).

Following a similar procedure as shown in Theorem 1, we get

$$u_{n-k}(t) = \int_0^t \frac{1}{k!} (t-s)^k T_A(t-s) Z(n, k) f(s) ds, \quad (k = 0, 1, 2, \dots, n-1). \quad (3.4)$$

According to (3.4), the solution of

$$\prod_{k=2}^n \left( \frac{d}{dt} - A \right) u(t) = f(t)$$

is

$$u_2(t) = \int_0^t \frac{1}{(n-2)!} (t-s)^{n-2} T_A(t-s) Z(n, n-2) f(s) ds.$$

To get the solution of the equation

$$\left( \frac{d}{dt} - B \right) \prod_{k=2}^n \left( \frac{d}{dt} - A \right) u(t) = f(t) \quad (3.5)$$

one need only to solve the equation

$$\left( \frac{d}{dt} - B \right) u_1 = u_2. \quad (3.6)$$

By Fubini's theorem and integration by parts, it is easy to see that

$$\begin{aligned} u_1(t) &= \int_0^t \frac{1}{(n-1)!} (t-\tau)^{n-1} (A-B)^{-1} T_A(t-\tau) Z(n, n-2) f(\tau) d\tau \\ &\quad - \int_0^t \int_\tau^t \frac{1}{(n-2)!} (s-\tau)^{n-2} T_A(s-\tau) \\ &\quad \times T_B(t-s) (A-B)^{-1} Z(n, n-2) f(\tau) ds d\tau. \end{aligned} \quad (3.7)$$

Since the operators  $A_j$ s are mutually commuted, without loss of generality we may assume that the same operator  $(\frac{d}{dt} - A_j)$  in (3.1) are put together such as (P1) and rewrite the differential equation in (3.1) as

$$\prod_{j=1}^{S_1} \left( \frac{d}{dt} - B_1 \right) \prod_{j=1}^{S_2} \left( \frac{d}{dt} - B_2 \right) \cdots \prod_{j=1}^{S_l} \left( \frac{d}{dt} - B_l \right) u = f. \quad (3.8)$$

We denote

$$W(t; j, B_l) = \sum_{j=0}^{S_l-1} \int_0^t \frac{1}{j!} (t-\tau)^j T_{B_l}(t-\tau) Z(j, l, B_l) g(\tau) d\tau \quad (3.9)$$

to be the solution of

$$\prod_{j=1}^{S_l} \left( \frac{d}{dt} - B_l \right) u(t) = g, \quad (3.10)$$

where the index  $j$  in  $W(t; j, B_l)$  denotes the counting number of steps in the iterative procedure for solving the problem from the beginning. Then the solution of (3.8) can be represented as

$$\begin{aligned} u(t) &= \sum_{j=0}^{S_1-1} \int_0^t \frac{1}{j!} (t-\tau)^j T_{B_1}(t-\tau) Z(n, j, B_1) f(\tau) d\tau \\ &\quad + \sum_{j=0}^{S_2-1} \int_0^t \frac{1}{j!} (t-\tau)^j T_{B_2}(t-\tau) Z(n, j, B_2) f(\tau) d\tau + \dots \\ &\quad + \sum_{j=0}^{S_l-1} \int_0^t \frac{1}{j!} (t-\tau)^j T_{B_l}(t-\tau) Z(n, j, B_l) f(\tau) d\tau. \end{aligned} \quad (3.11)$$

Simplifying the notation we denote the coefficient of (3.11) as

$$\begin{aligned} Z(n, j, B_1) &= Z(n, k_1) \text{ for } 0 \leq j = k_1 \leq S_1 - 1; \\ Z(n, j, B_2) &= Z(n, S_1 + k_2) \text{ for } 0 \leq j = k_2 \leq S_2 - 1; \\ &\vdots \\ Z(n, j, B_l) &= Z\left(n, \sum_{j=1}^{l-1} S_j + k_l\right) \text{ for } 0 \leq j = k_l \leq S_l - 1. \end{aligned}$$

This theorem will be proved as long as one find the explicit form of  $Z(n, j, B_l)$ . One may follow the procedure in Theorem 1 to find the matrix  $M_n$ . As in Theorem 1, we begin with the special case that all  $A_i$ s are equal to  $A$ , then combine the results of distinct  $B_j$ s part to get the general form  $M_n$ . When all  $A_i$ s are equal to  $A$ , the equation is of the form

$$\prod_{k=1}^n \left( \frac{d}{dt} - A \right) u = f,$$

and then the solution of this equation is

$$u(t) = W(n, t, A) = \sum_{j=0}^{n-1} \int_0^t \frac{1}{j!} (t-s)^j T_A(t-s) Z(n, j) f(s) ds. \quad (3.12)$$

One may get (1.8) by continuously differentiating (3.12). The first derivative of (3.12) gives

$$\begin{aligned} u'(t) &= \sum_{j=1}^{n-1} \int_0^t \frac{1}{(j-1)!} (t-s)^{(j-1)} T_A(t-s) Z(n, j) f(s) ds \\ &\quad + \sum_{j=0}^{n-1} \int_0^t \frac{1}{j!} (t-s)^j A T_A(t-s) Z(n, j) f(s) ds + Z(n, 0) f(t). \end{aligned} \quad (3.13)$$

The initial condition  $u'(0) = 0$  implies  $Z(n, 0) f(0) = 0$ . Since  $f(0) \neq 0$ , it enforces  $Z(n, 0) = 0$ . Continuing this procedure, one may get

$$\begin{aligned} u^{(i)}(t) &= \sum_{j=i}^{n-1} \int_0^t \frac{1}{(j-i)!} (t-s)^{j-i} T_A(t-s) Z(n, j) f(s) ds \\ &\quad + \sum_{j=i-1}^{n-1} \int_0^t \frac{1}{(j-(i-1))!} (t-s)^{(j-(i-1))} \\ &\quad \times C_1^i A T_A(t-s) Z(n, j) f(s) ds + \dots \\ &\quad + \sum_{j=i-k}^{n-1} \int_0^t \frac{1}{(j-(i-k))!} (t-s)^{(j-(i-k))} \\ &\quad \times C_k^i A^k T_A(t-s) Z(n, j) f(s) ds + \dots \\ &\quad + \sum_{j=0}^{n-1} \int_0^t \frac{1}{j!} (t-s)^j C_i^i A^i T_A(t-s) \\ &\quad \times Z(n, j) f(s) ds + Z(n, i-1) f(t) \\ &\quad + C_1^{i-1} A Z(n, i-2) f(t) + \dots + C_{i-1}^{i-1} A^{i-1} \\ &\quad \times Z(n, 0) f(t) \quad \text{for } 1 \leq i \leq n \end{aligned}$$

and

$$Z(n, i-1) + C_1^{i-1} A Z(n, i-2) + \dots + C_{i-1}^{i-1} A^{i-1} Z(n, 0) = 0 \quad \text{for } 1 \leq i \leq n-1. \quad (3.14)$$

Finally, one can put  $u(t)$ ,  $u^{(i)}(t)$  for  $1 \leq i \leq n$  into (3.1) to get

$$\{Z(n, n-1) + C_1^{n-1} A Z(n, n-2) + \dots + C_{n-1}^{n-1} A^{n-1} Z(n, 0)\} f = f. \quad (3.15)$$

Put (3.14) and (3.15) together and let  $\vec{h} = (0, 0, \dots, I)^T$ , where  $I$  is the identity operator on  $C^1([0, T]; X) \cap (\cup_{i=1}^n C[0, T]; [D(A_i)])$ . Then one can write them in the matrix form

$$M_n^{-1}\vec{h} = \vec{z}.$$

One can apply the superposition principle to get the solution of (3.1) with nonzero initial data. It is the sum of the solutions of (1.1) and (3.1) with zero initial data. We summarize this result as follows.

**COROLLARY 3.**

*Under the hypotheses of Theorems 1 and 2, eq. (3.1) with nonzero initial data has a solution  $u(t)$  which is represented as*

$$\begin{aligned} u(t) = & \sum_{k_1=0}^{S_1-1} \frac{t^{k_1}}{k_1!} T_{A_1}(t) y(n, k_1) + \sum_{k_2=0}^{S_2-1} \frac{t^{k_2}}{k_2!} T_{B_2}(t) y(n, S_1 + k_2) + \cdots \\ & + \sum_{k_i=0}^{S_i-1} \frac{t^{k_i}}{k_i!} T_{B_i}(t) y \left( n, \left( \sum_{j=1}^{i-1} S_j \right) + k_i \right) + \cdots \\ & + \sum_{k_1=0}^{S_1-1} \int_0^t \frac{1}{k_1!} (t-s)^{k_1} T_{B_1}(t-s) Z(n, k_1) f(s) ds \\ & + \sum_{k_2=0}^{S_2-1} \int_0^t \frac{1}{k_2!} (t-s)^{k_2} T_{B_2}(t-s) Z(n, S_1 + k_2) f(s) ds + \cdots \\ & + \sum_{k_i=0}^{S_i-1} \int_0^t \frac{1}{k_i!} (t-s)^{k_i} T_{B_i}(t-s) Z \left( n, \sum_{j=1}^{i-1} S_j + k_i \right) f(s) ds. \end{aligned}$$

#### 4. Applications

*Example 1.* We consider the following initial value problem

$$\begin{cases} u_{tt}(t, x) + a_1 u_{tx}(t, x) + a_2 u_{xx}(t, x) = f(t, x), & (t, x) \in [0, T] \times R; \\ u(0, x) = \phi_1(x), & x \in R; \\ u_t(0, x) = \phi_2(x), & x \in R; \end{cases} \quad (4.1)$$

where  $a_1, a_2$  are given constants. Let  $E = L^2(R)$ ,  $D(A^k) = W_2^k(R)$  and  $A^k f = \frac{d^k}{dx^k} f$ ,  $k = 1, 2$  for every  $f \in D(A^k)$ . Under these notations, (4.1) is equivalent to the following initial value problem:

$$\begin{cases} U''(t) + a_1 A U'(t) + a_2 A^2 U(t) = F(t), & t \in [0, T]; \\ U(0) = \phi_1(x), & x \in R; \\ U'(0) = \phi_2(x), & x \in R; \end{cases} \quad (4.2)$$

where  $U(t) \in L^2(R)$  and  $F(t) \in L^2(R)$  satisfy  $(U(t))x = u(t, x)$ ,  $(F(t))x = f(t, x)$  for all  $(t, x) \in [0, T] \times R$ . It is well-known that  $A$  generates a  $C_0$ -semigroup  $\{T(t)\}_{t \geq 0}$  on the Banach space  $E$  which satisfies  $(T(t)f)(x) = f(x+t)$  for all  $f \in L^2(R)$  and for all  $(t, x) \in$

$[0, T] \times R$  (see for e.g. Ch. 22, item 22.5 of [7]). Moreover,  $T(t)$  can be extended to a  $C_0$ -group. It is easy to see that

$$\frac{d^k}{dt^k} A^{2-k} h(t) = A^{2-k} \frac{d^k}{dt^k} h(t), \quad k = 1, 2 \text{ for any } h \in (C^2[0, T] : D(A^2)).$$

If the characteristic equation  $P(z) = z^2 + a_1 z + a_2 = 0$  of (4.2) has a root  $z_1$  with multiplicity 2, then  $z_1 A$  generates a  $C_0$ -semigroup  $T_1(t)$  which satisfies  $(T_1(t)f)(x) = f(x + z_1 t)$  for all  $f \in L^2(R)$  and for all  $(t, x) \in [0, T] \times R$ .

If  $\phi_1 \in D(A^3)$  and  $\phi_2 \in D(A^2)$ , then (4.2) can be rewritten as

$$\begin{cases} \left( \frac{d}{dt} - z_1 A \right)^2 U = F(t); \\ U(0) = \phi_1(x); \\ U'(0) = \phi_2(x). \end{cases} \quad (4.3)$$

By Corollary 3, (4.3) has a solution of the form

$$\begin{aligned} U(t) &= T_1(t)y(2, 0) + tT_1(t)y(2, 1) + \int_0^t T_1(t-s)Z(2, 0)F(s) ds \\ &\quad + \int_0^t (t-s)T_1(t-s)Z(2, 1)F(s) ds, \end{aligned}$$

where  $y(2, 0)$ ,  $y(2, 1)$  and  $Z(2, 0)$ ,  $Z(2, 1)$  satisfy

$$\begin{bmatrix} 1 & 0 \\ z_1 A & 1 \end{bmatrix} \begin{bmatrix} y(2, 0) \\ y(2, 1) \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}; \quad \begin{bmatrix} 1 & 0 \\ z_1 A & 1 \end{bmatrix} \begin{bmatrix} Z(2, 0) \\ Z(2, 1) \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

This implies that

$$U(t) = T_1(t)\phi_1 + tT_1(t)(\phi_2 - z_1 A\phi_1) + \int_0^t (t-s)T_1(t-s)F(s) ds.$$

Thus, (4.1) has a solution

$$\begin{aligned} u(t, x) &= \phi_1(x + z_1 t) + t\phi_2(x + z_1 t) - z_1 t\phi_1'(x + z_1 t) \\ &\quad + \int_0^t (t-s)f(s, z_1(t-s) + x) ds. \end{aligned}$$

*Example 2.* We consider the following initial boundary value problem

$$\begin{cases} \frac{\partial^2}{\partial t^2}(u(t, x)) + b_1 \frac{\partial}{\partial t}(\Delta u(t, x)) + b_2 \Delta^2 u(t, x) = f(t, x), & (t, x) \in (0, T) \times \Omega; \\ u(0, x) = \Psi_1(x), & x \in \Omega; \\ \frac{\partial}{\partial t}(u(0, x)) = \Psi_2(x), & x \in \Omega; \\ u(t, x) = 0, & (t, x) \in [0, T] \times \partial\Omega; \end{cases} \quad (4.4)$$

where  $b_1, b_2$  are constants and  $\Omega \subset \mathbb{R}^n$  is a bounded domain with smooth boundary. Let  $E = L^2(R)$ ,  $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$  and  $Av = \Delta v$ ,  $\forall v \in D(A)$ . Then (4.4) can be written as

$$\begin{cases} \widetilde{U}_{tt}(t) + b_1 A \widetilde{U}_t(t) + b_2 A^2 \widetilde{U}(t) = \widetilde{F}(t); \\ \widetilde{U}(0) = \Psi_1, \quad \widetilde{U}_t(0) = \Psi_2. \end{cases} \quad (4.5)$$

Through a simple calculus, one may get  $\frac{d^k}{dt^k} A^{2-k} h(t) = A^{2-k} \frac{d^k}{dt^k} h(t)$ ,  $k = 1, 2$  for any  $h \in C^2([0, T]; D(A^2))$ . Pazy (p. 211 of [8]) shows that  $A$  is the generator of an analytic  $C_0$ -semigroup  $\{\widetilde{T}(t)\}_{t \geq 0}$ . It is easy to show that (see e.g., p. 104 of [8]) for any  $g \in L^2(\Omega)$ , the initial boundary value problem

$$\begin{cases} \frac{\partial}{\partial t}(y(t, x)) = \Delta y(t, x), & (t, x) \in (0, T) \times \Omega; \\ y(0, x) = g(x), & x \in \Omega; \\ y(t, x) = 0, & (t, x) \in [0, T] \times \partial\Omega; \end{cases} \quad (4.6)$$

has a unique solution  $y(t, x) = (\widetilde{T}(t)g)(x)$ . However, the initial boundary value problem (4.6) can be solved by separation of variables method. Its solution can be represented as

$$y(t, x) = \sum_{k=0}^{\infty} \alpha_k e^{\lambda_k t} w_k(x),$$

where  $0 > \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$  are the eigenvalues of the Laplace equation with Dirichlet boundary conditions and  $w_k(x)$  are the corresponding normalized eigenfunctions. Thus

$$(\widetilde{T}(t)g)(x) = \sum_{k=0}^{\infty} \beta_{k,g} e^{\lambda_k t} w_k(x).$$

If the characteristic equation  $P(z) = z^2 + b_1 z + b_2$  of (4.5) has a root  $\widetilde{\alpha}_1$  with multiplicity 2, then  $\widetilde{\alpha}_1 A$  generates a  $C_0$ -semigroup  $\{\widetilde{T}_1(t)\}_{t \geq 0}$  and the solution of (4.5) can be represented as

$$(\widetilde{T}_1(t)g)(x) = \sum_{k=0}^{\infty} \beta_{k,g} e^{\lambda_k \widetilde{\alpha}_1 t} w_k(x) \text{ for } g \in L^2(\Omega), (t, x) \in (0, T) \times \Omega.$$

As Example 1, this implies that

$$U(t) = T_1(t)\Psi_1 + tT_1(t)(\phi_2 - \widetilde{\alpha}_1 A\phi_1) + \int_0^t (t-t_2)T_1(t-t_2)\widetilde{F}(t_2) dt_2.$$

Furthermore, if  $\Psi_1 \in D(A^3)$ ,  $\Psi_2 \in D(A^2)$ ,  $\widetilde{F} \in C^2([0, T]; L^2(\Omega))$ ,  $\widetilde{F}(t) \in D(A^2)$  for all  $t$  in  $(0, T)$  and  $\widetilde{F}(0) \in D(A)$ , then the solution of (4.4) can be represented as

$$\begin{aligned} u(t, x) = & \sum_{k=0}^{\infty} [\beta_{k, \Psi_1} + t\beta_{k, \Psi_2 - \widetilde{\alpha}_1 \Delta \Psi_1}] e^{\lambda_k \widetilde{\alpha}_1 t} w_k(x) \\ & + \int_0^t (t-s) \left[ \sum_{k=0}^{\infty} \beta_{k, \widetilde{F}(s, x)} e^{\lambda_k \widetilde{\alpha}_1 (t-s)} w_k(x) \right] ds. \end{aligned}$$



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